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EXPERIMENTAL TEACHING OF MATHEMATICAL LOGIC IN THE ELEMENTARY SCHOOL.

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STUDY OBJECTIVES WERE DEVELOPED AROUND THE IDEA THAT DELIBERATE AND PLANNED TEACHING OF FORMAL SYMBOLIC LOGIC WILL BENEFIT THE STUDENT AT ANY AGE BY ENLARGING HIS SCOPE AND PROVIDING FOR A DEEPER AND MORE PENETRATING STUDY OF MATHEMATICS AND OTHER DEDUCTIVELY FORMULATED DISCIPLINES. IT WAS HYPOTHESIZED THAT THE ABLE STUDENT IN ELEMENTARY SCHOOL IS CAPABLE OF ACQUIRING THE SKILLS TO A SIGNIFICANT DEGREE AND WITHIN A REASONABLE TIME. APPROXIMATELY 500 ACADEMICALLY TALENTED FIFTH- AND SIXTH-GRADE STUDENTS PARTICIPATED IN AN EXPERIMENTAL 2-YEAR PROGRAM IN 13 SCHOOLS. IT WAS FOUND THAT--(1) THE UPPER QUARTILE OF ELEMENTARY SCHOOL STUDENTS CAN ACHIEVE CONCEPTUAL AND TECHNICAL MASTERY OF ELEMENTARY MATHEMATICAL LOGIC, (2) MASTERY OF THE SUBJECT MATTER CAN BE ACCOMPLISHED IN AN AMOUNT OF TIME COMPARABLE TO THAT NEEDED BY COLLEGE STUDENTS, BUT DISTRIBUTED OVER A GREATER PERIOD OF TIME AND WITH CONSIDERABLY MORE DIRECT TEACHER SUPERVISION, AND (3) THE MORE DEDICATED AND ABLE ELEMENTARY-SCHOOL TEACHERS CAN BE GIVEN SUFFICIENT TRAINING IN 5 OR 6 SEMESTER HOURS OF INSTRUCTION TO TEACH CLASSES IN ELEMENTARY MATHEMATICAL LOGIC. (JM)

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EXPERIMENTAL TEACHING OF MATHEMATICAL LOGIC IN THE ELEMENTARY SCHOOL

Cooperative Research Project No. D-005

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4. The problem.

For many years, but especially since the successful launching of Sputnik I, there has been a great concern to upgrade the mastery of mathematics by students before they arrive at the college level. A typical concern is that the student be ready to use the calculus in college freshman physics.

Initially this concern was interpreted as simply requiring the revamping of the high-school mathematics curriculum so that the concepts and skills of the calculus and analytic geometry could be introduced to the high-school senior. The Commission on Mathematics of the College Entrance Examination Board recommended this in its 1958 Mathematics for Today. It also recommended that the high-school student be introduced to basic concepts of mathematics such as the different kinds of numbers, inequalities, sets, variables, functions, relations; and the nature and role of deductive reasoning in algebra as well as in geometry.

In recent years no one has supposed that high-school students are incapable of grasping such concepts. The problem was viewed as simply that of taking the student at the level of achievement at which he commonly enters high-school and raising him to the demanded level during the high-school years.

This was attempted along the two lines of allotting more time for the study of mathematics in the high-school curriculum and of trying to raise the efficiency of mathematics instruction by eliminating unessential topics and eliminating the traditional concepts of mathematics as divided into separate branches to be studied one at a time. There is new emphasis on fundamental mathematical concepts that cut across the older divisions.

Thus, mathematics is coming to be taught as a set of fundamental principles and their deductive consequences and applications. It is hoped, and proving to be the case, that the study of mathematics at any stage is thus made more efficient and successful; and that each stage more adequately prepares the student for future stages, thus raising the efficiency of the total process.

In more recent years a third line of attack has been getting attention. This is to bring the student to an improved level of mathematical achievement before he enters high school. This is done both by introducing topics previously not taught before high school and by presenting more of the foundations of what is taught. The student is led to understand why as well as how desired results can be obtained.

The student is taught to see mathematics as sets of deductive consequences from sets of possible assumptions.

The specific problem that served as the focus of the present project was the exploration of the possibility of teaching deductive techniques to gifted elementary-school children. In particular, the project has attempted to find out how easy or difficult, as well as how appropriate, is the learning of the elements of mathematical logic by fifth- and sixth-grade students.

5. Objectives and hypotheses.

This working out of deductive consequences involves the theory of logical validity and ability to recognize invalidity or fallacy. The recognition of possible assumptions involves the theory of logical consistency. The theories of validity and invalidity and of consistency and inconsistency are the subject matter of formal mathematical logic and are basic to the deductively formulated disciplines of mathematics and the sciences. Yet, in the usual curriculum, the development of children's reasoning abilities is left largely to incidental learning.

The point of view of this project is that deliberate and planned teaching of formal symbolic logic will benefit the student at any age by enlarging his scope and providing for a deeper and more penetrating study of mathematics and other deductively formulated disciplines. The hypothesis projected is that the able student in elementary school is capable of acquiring these skills to a significant degree and within a reasonable time.

The encouragement of critical thinking is generally acknowledged to be among the legitimate objectives of our schools. Yet there is little

indication of a generally accepted translation of this goal into behavioral terms. If children are to be taught to think critically it is necessary that some clear notion of the component skills guides the selection of specific content of classroom activities. Among the skills essential to critical thought are abilities to make logically correct inferences, to recognize fallacies, and to recognize inconsistencies among statements. If the ability to derive logical conclusions from given evidence, data, or premises is a factor in critical thinking, then instruction designed to develop or improve these skills must be a concern of the school.

This experimental project assumes that deliberate instruction in precise and systematic ways of thinking may contribute significantly to the child's skill in critical thinking, that children in elementary school can be taught to make logical deductions, and that at least the more able among them may profit from a formal treatment of logic.

In general terms the aim of the program is to deepen and extend the mathematical experiences of the able elementary school child at the broadest level of mathematics, the level of methodology and the theory of proof. The approach is through a study of modern mathematical logic, in particular that portion of it which is concerned with the theory of logical inference or the theory of deduction. In addition the theory of logical consistency is also studied.

In modern times logic has become a deep and broad subject. Only in recent years have the systematic relations between logic and mathematics been established and a completely explicit theory of inference formulated which is adequate to deal with all the standard examples of deductive reasoning in mathematics and the empirical sciences.

The concept of axioms and the deduction of theorems from axioms is at the heart of all modern mathematics. The purpose of this project is to introduce the academically gifted, elementary-school child to modern mathematics and mathematical methods at a level which is rigorous but simple enough in presentation and context to permit comprehension by elementary- and secondary-school students.

Specifically, the objective of the project is to experiment on the feasibility of teaching mathematical logic to classes of academically

talented fifth- and sixth-grade children. Thus it is considered a program in enrichment for the more able mathematics student and a supplement to the elementary-school mathematics curriculum. Questions of particular interest involve the capacity of children of the fifth- and sixth-grade age level to do the kinds of deductive proof which are characteristic of modern mathematics, specific factors of difficulty, and the possible transfer of skills of analysis and correct reasoning to other subject matter areas.

In the study of logic, the child is introduced to a way of using language and ideas precisely. The emphasis is upon clarity and precision in thinking, upon rigor and consistency. The student is learning to recognize the logical structure of English sentences and of discourse. The introduction of the concept of form and structure in language, and analysis of structure are integral parts of the child's study. Thus it is clear that while this project is considered a supplement to the program in mathematics in the elementary school, the application of the concepts and skills learned is not limited to this subject. The principles of logical inference are universally applied in every branch of systematic knowledge.

Implementing these objectives immediately generates problems of teacher preparation. A secondary objective then is the training of teachers. It is necessary to learn how much training is necessary to enable the secondary school teacher to adequately present the formal instruction in logic to the fifth- and sixth-grade children.

6. Related research.

Research which bears upon the capacities of children of particular ages to reason logically is scarcely definitive. Inhelden and Piaget (1958) have concluded that development of capacity for "hypothetical" reasoning or formal aspects of logic begins at about age eleven. However, a recent study by Hill (1960) indicates that children of ages six, seven, and eight have a considerable intuitive grasp of many of the principles of logical inference and can demonstrate this grasp in reasoning from hypothetical premises. Her results also indicate that

simple demonstrations of correct deductions improve children's performance in the recognition of valid inference.

This research by Dr. Hill was carried out under the Office of Education's Cooperative Research Program, contract SAE 8406. A study by Hyram (1957) has also suggested that improvement in children's inferences may be brought about by direct instruction in logical rules.

7. Procedure.

A. Pilot study 1960-1962

A pilot study in the teaching of mathematical logic to a group of selected fifth-grade students was begun at Clifford Public School in Redwood City, California, in the autumn of 1960. The class consisted of 25 students selected from the several fifth-grade classes in the school on the basis of their arithmetic achievement scores. The children had been given the arithmetic test during the seventh month of their fourth-grade year and only students who had scored at the sixth-grade level or above at that time were selected for the logic program. The class comprised about 27 percent of the fifth-grade students in the school.

The class met three days per week throughout the school year for sessions of 30 to 35 minutes. The class was taught in 1960-1961 by Dr. Shirley Hill and in 1961-1962 by Mr. Frederick Binford, both of the project staff.

Textbook material, composed of explanatory material and practice exercises, was prepared for use by the class.

B. Teacher preparation.

During the summer of 1961 an intensive four-week course was given to ten teachers and three of the school principals. This class used the materials developed for the up-coming classes of elementary-school students. Another 14 teachers took a night-school course in the spring of 1962.

In the fall of 1962, a second course was given. Seventeen of those teachers who had taken the first course were enrolled while teaching their own first- or second-year class in logic to fifth- or sixth-grade children.

Again in the spring of 1963 a teacher preparation class was offered. Twenty-six teachers enrolled. Those teachers will be offered a second course in the fall of 1963.

C. Experimental Classes.

1961-1962. Twelve classes with a total of 350 fifth-grade students began their study of logic in the fall of 1961. These classes represented 11 different schools in six school districts in the San Francisco Bay Area. In all but two schools, approximately 25 to 30 percent of the fifth graders were selected by the school administration to join the class on the basis of general ability and high achievement in mathematics. In two schools, all of the fifth graders were given logic. In both of these schools it was found necessary within a few months to divide the logic classes into fast and slow sections.

Most of the classes met three times a week for about 30 minutes. However, a few of them met more often or for longer sessions. Each of the classes was taught by a classroom teacher from the staff of the school with the exception of Stanford Elementary School whose class was taught by Mr. Binford of the Project staff.

The work in the experimental classes largely followed the sequence presented in the text book. No effort was made to maintain a uniform rate of progress in all classes but each teacher proceeded at the pace that seemed appropriate to that class. During the year these classes completed from 115 to 195 pages of the text, Mathematical Logic for the Schools by Patrick Suppes and Shirley Hill.

1962-1963. Eleven classes of sixth graders consisting of 215 students who had their first year of logic in fifth grade (1961-1962) continued with their second year of logic study. Most of the classes were reviewing throughout the autumn. By June, these classes had completed between 162 and 284 pages.

In addition, 12 new classes of 269 fifth-grade students began their first year of logic study and completed between 117 and 183 pages of the text. One class began late in February with 18 pupils and completed 60 pages of the text.

Table 1 shows the schools, teachers, number of pupils, pages of text covered and tests taken by the various classes during the two years.

Table 1. Record of classes, including number of pupils, pages of text covered and tests taken.

	First Year of Logic				Second Year		
	Year class began	Number of students	Page reached	Tests	No. of stud.	Page reached	Tests
Green Gables, Palo Alto							
Mrs. Nancy Simon	1961	21	195 (226)	I,II,III	14	285	I,II-1,2
Miss Martha Logan	1962	23	183	I			
Grant, Los Altos							
Mr. Don Eldridge	1961	37	154 (183)	I,II	20 17	260 183	I,II-1,2 I,II-1
Mr. Wm. Howe	1961	31	86,(124) 125-138	I	This was a class of non-gifted students, one year only.		
Mr. Dale Doty	2/1963	18	60				
Clifford, Redwood City							
Mr. Fred Binford (Pilot)	1960	25	160		18	280	
Mr. Carl Sorem	1961	26	154	I,II	16	238	I,II-1
Mrs. Eleanor Cooper	1962	26	183	I,II-1			
Franklin, Redwood City							
Mrs. Geraldine Kirchgatter	1961	33	138 (162)	I	22	230	I,II-1
Luther School, Santa Clara							
Mrs. Betty Ortez	1961 1962	15 20	138 183	I	8	236	
Lincoln, Redwood City							
Mrs. Jordis Scarborough	1961 1962	29 15	138 137	I I	15	243	I,II-1,2

Table 1. (cont.)

	First Year of Logic				Second Year		
	Year class began	Number of students	Page reached	Tests	No. of stud.	Page reached	Tests
Roosevelt, Redwood City							
Miss Vi Susan	1961	30	154	I,II	25	266	I,II-2
Mr. Ross Pfann	1962	35	183	I,II-1			
Highlands, Milbrae							
Mrs. Katherine Nelson	1961	48 (2 classes)	154	I,II	27	270	I,II-1,2
Mr. Perry Broz	1962	29	183	I,II-1			
Sunshine Gardens, So. San Fran.							
Mrs. Frances Grimes	1961	31	138	I	18	165	I
Miss Donna Olson	1962	26	145	I			
Lakeshore, San Mateo							
Mr. Arthur Kohn	1961	20	101 125-138	I	8	162	I
	1962	15	117	I			
Stanford Elementary, Stanford							
Mr. Fred Binford	1961	29	154	I,II	20	241	I,II-1
Mr. Guy Noerdinger	1962	24	183				
Dilworth, San Jose							
Mr. Bernard La Casse	1962	16	183	I,II-1			
Mr. Richard Burchby	1962	18	183	I,II-1			
North Davis, Davis							
Mrs. Marla Fisher	1962	24	183	I,II-1			

It should be pointed out that the text used in 1961-1962 was revised and expanded and some of the materials were arranged in different order during the summer of 1962. The numbers in parentheses in the column listing the page reached during the 1961-1962 academic year, indicate the page equivalence in the 1962-1963 edition of the text.

D. Control groups.

The control groups were two Stanford University logic classes taking logic at the customary age and educational level. These classes included college undergraduates and graduates with the mode being between the sophomore and junior year of college. The text prepared for the elementary-school students was used. These classes met in the autumn quarters of 1961 and 1962.

It can be fairly assumed that the native ability of the elementary-school children selected for the program is comparable to that of the Stanford University students.

The 1962 class was taught by Professor Patrick Suppes, the Project director.

E. Testing program.

A series of tests was prepared to be given to the various classes and to the control groups whenever they had reached certain points in the text. The 1961-1962 forms of the tests were not for objective scoring. The 1962-1963 forms were for objective scoring. In both instances the teachers scored the tests of their own classes. Samples of the 1962-1963 tests are appended.

The 1961-1962 examinations and the page of the text were as follows:

	Test	Page in 1961-1962 edition	Page in 1962-1963 edition
1961-1962 forms	I	83	(111)
	II	154	(183)
1962-1963 forms	I.		111
	II. part 1		183
	part 2.		243

F. Curriculum material.

During the summer of 1961 materials developed by Professor Suppes and Dr. Hill for the pilot class in 1960-1961 were extended and revised with suggestions from Mr. Binford. This text material was used for the classes in 1961-1962 and is titled Mathematical Logic for the Schools.

Again in the summer of 1962 on the basis of the experience of teachers and children, the text was revised, further exercises added, some changes in rules and form were made, and some of the materials were arranged in different order.

In the summer of 1961, Mr. Binford and Dr. Hill prepared a manual giving solutions for all of the exercises in Mathematical Logic for the Schools. In the summer of 1962 Mr. Binford revised this manual to accompany the revised text. At the same time several comments, suggestions, and explanations were added for the teacher.

The Table of Contents of the text material, as revised in 1962, was as follows:

Chapter 1	Symbolizing Sentences	1
	1.1 Sentences	1
	1.2 Sentential Connectives	2
	1.3 The Form of Molecular Sentences	6
	1.4 Symbolizing Sentences	10
	1.5 The Sentential Connectives and Their Symbols	14
	Or	16
	Not	18
	If...then...	22
	1.6 Grouping and Parentheses	24
	The Negation of a Molecular Sentence	31
	1.7 Elimination of Some Parentheses	36
	1.8 Summary	38
Chapter 2	Logical Inference	46
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	2.2 Rules of Inference and Proof	47
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Proofs 51
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More on Negation 61
Adjunction and Simplification 64
Disjunction as Premises 66
Modus Tollendo Ponens 68
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2.5 Further Rules of Inference 83
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Law of Hypothetical Syllogism 87
Law of Disjunctive Syllogism 90
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Chapter 3 Truth and Validity 112
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Disjunction 115
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Chapter 4 Truth Tables 163
4.1 Truth Tables 163
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Chapter 5 Terms, Predicates and Universal Quantifiers 184
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5.2 Terms 187
5.3 Predicates 189
5.4 Common Nouns as Predicates 191
5.5 Atomic Formulas and Variables 195
5.6 Universal Quantifiers 201
5.7 Two Standard Forms 207

Chapter 6 Universal Specification and Laws of Identity 214
6.1 One Quantifier 214
6.2 Two or More Quantifiers 226
6.3 Logic of Identity 234
6.4 Truths of Logic 239

Chapter 7 A Simple Mathematical System: Axioms for Addition 244
7.1 Commutative Axiom 24[illegible]
7.2 Associative Axiom 248
7.3 Axiom for Zero 257
7.4 Axiom for Negative numbers 260

Chapter 8 Universal Generalization 266
8.1 Theorems with Variables 266
8.2 Theorems with Universal Quantifiers 270

G. Supervision.

The Project staff has given the teachers regular supervision. In 1961-1962 regular monthly meetings were held with the teachers. In addition, a member of the project staff visited each class about every six weeks, observing part of the time, teaching part of the time and conferring with the teacher.

During the autumn of 1962 the teachers were taking the second course and thus meeting with the Project staff every week. During the remainder of the year, a member of the staff visited each class and conferred with the teacher about every three weeks.

H. Special class, summer 1963.

A special follow-up class was offered to the most successful of the elementary school children who, upon completing sixth grade, had studied logic for two years. The children were selected upon the recommendation of their teachers and by a special examination given the candidates. Thus a class of 26 students was organized. Joining the class was entirely voluntary for the children. The parents provided transportation to Stanford University campus. In several cases the student or his whole family altered summer plans in order to join. The class ran for four weeks, meeting five days a week for sessions one full hour in length. In addition the students did outside assignments. Mr. Binford taught the classes. New materials were used which had been prepared to follow the curriculum materials prepared for the regular course. These materials presented the logic of equivalence and developed the theory of multiplication and division from standard axioms.

At the end of four weeks we announced a voluntary two-weeks extension of the special class. Ten of the students continued. During these two weeks they studied the algebra of sets and some elementary topology. From nine axioms and three definitions they proved 45 theorems of set theory. Then with four axioms and four definitions of point-set topology they proved 20 additional elementary theorems of topology. The list of axioms and theorems is appended.

8. Analyses of test data and other findings.

Our examinations of achievement in the subject matter of mathematical logic indicate that the level of accomplishment of the elementary-school students is comparable to that of a college class in mathematical logic although their pace is slower. However, it should be pointed out that the college students put in about ten times as many hours each week on logic as do the elementary-school students. So, although the college students achieve about as much in four weeks as do the elementary-school students in an academic year, their achievements per hour are comparable. It is suggested that the two groups, experimental and control, may be considered comparable in that both represent a selected group in terms of the general population. It is understood that this comparison of achievement is not at comparable time periods but at comparable points of accomplishment in covering the systematically organized body of course content.

Three tests were given to the college group of 186 students in the fall of 1961. The first of these tests was also given to all eleven fifth-grade classes, 260 students. The test was on the recognition of the logical structure of sentences and early work in symbolic deduction. A few very low scores pulled the mean well below the median. For the college group: mean 91, median 97. For the fifth grades: mean 76, median 83.

Seven fifth-grade classes (164 students) took the second test. This tested mostly the symbolic manipulations involved in deductions. The results were for the college group: mean 70, median 72; for the fifth-grade group: mean 67, median 77. The mean scores of the fifth-grade classes were: 92, 87, 79, 78, 76, 49, and 36--five of the seven above the college mean.

Table 2. Summary Test Results for 1961-62

	POSSIBLE SCORE - 100			
Test I	N	MEAN	MEDIAN	RANGE
College Students	186	91	97	8-100
Fifth-Grade Students	260	76	83	17-100
Test II				
College Students	189	70	72	19- 99
Fifth-Grade Students	164	67	77	6-100

These test results suggest that the symbolic manipulations characteristic of formal proofs are relatively easier for elementary school children than the problem of recognizing the logical form or structure of sentences in ordinary English.

The testing program continued in 1962-1963. Another control group of college students was tested. As the experimental classes of students in sixth grade advanced, they took the same tests. At the same time beginning classes of fifth grades also joining the program were tested.

The 1962-1963 edition of the tests was made out in multiple-choice form with answers to be transferred to a standard answer sheet. This was done to allow objective and rapid scoring.

In Test I, Sections A, B, and C covered the translation of English into logical symbols, the construction of negations, and classification of different logical forms for sentences. Section D covered immediate application of rules of inferences to one-step derivations. Section E covered logical vocabulary and Sections F and G covered the making of multiple-step derivations.

The following table summarizes the finding on Test I in 1962-1963.

Table 3. Results of Test I, 1962-1963

The section of the test		By sections of the test				Total test
		A,B,C	D	E	F,G	
number of items in the section		26	21	12	26	85
Control group, college students scores	mean	22.96	20.25	11.05	24.75	79.0
	median	24	21	11	26	81
	range	26-15	21-7	12-2	26-0	85-24
Experimental groups, elementary school students scores	mean of entire group	19.0	17.0	10.1	22.9	69.5
	average of the medians of the separate classes	19.9	17.5	10.3	24.2	71.1
	range class means	22-15.5	18.9-13.5	11.5-8.5	25.8-18.7	77.1-59.0
	range of all students	25-5	21-7	12-5	26-4	84-32
Ratios of elementary school mean scores to college	all elementary school students to college	.83	.83	.90	.94	.88
	mean score of the best elementary school class to college	.96	.89	1.09	1.04	.975

The elementary school children as compared to the college control did relatively better on carrying out the pure symbolic logical deductions (Sections F and G). Similar results were obtained in 1961-1962. Over-all the elementary-school children scored about 88 percent as high as the college control group. Except for question 41, which none of the elementary group got correct, the ranges for each section of Test I were nearly identical for the two groups--perfect test results were obtained in both groups and the best of the experimental classes was quite comparable to the college class.

Test II was written in two parts, to be given when the classes were ready for each part. The section A of part one covered deductions in multiple-step problems. It contained 28 items. The sections B, C, D, E tested on proofs of invalidity and of consistency and inconsistency. They contained 11 items triple weighted. Ten of the experimental classes totaling 196 students completed the first part of the test. Three of these classes with a total of 60 students also completed the second part, which covered translation from English into the symbolism of predicate logic and the carrying out of derivations as given in Chapter 7 (see Table of Contents given above). This part of the test contained 41 items.

In part one of the test the children averaged about 83 percent as high as the college control group, the best class almost exactly matching the college mean. The ranges for experimental and control groups were identical, several perfect test results being obtained in each group.

The three experimental classes completing all of Test II averaged 87 percent as high as the control class. One college student handed in a perfect paper, one elementary-school student lost only two points. The lowest scores were 31 and 34 respectively. The following table summarizes the results on Test II.

Table 4. Results of Test II, 1962-1963

		Part 1			Total of parts 1 and 2
		A	B-E	Total of part 1	
Possible score		28	33	61	102
College control group	mean	27.3	25.8	53.1	80.8
	median	28	27	55	82
	range	28-12	33-6	61-18	102-31
Elementary school, experimental group	mean	25.2	19.7	44.2	70.2*
	range	28-12	33-3	61-19	100-34*
Ratio of means: experimental to control		.92	.76	.83	.87*

*These figures are based on the three classes totaling 60 students that took both parts of the test.

Comparisons between the 1961-1962 and the 1962-1963 test results require interpretation. Of particular interest is the Test II results where in 1961-1962 the elementary school median was 7 percent above the college median, while in 1962-1963 it was some 15 percent below. The teachers administering the later tests reported that the students had difficulty handling the mechanics of the objective tests and were often confused by the multiple choices. One teacher whose class had a median of five points out of 15 on the last section of Test II later gave them the same problems to work out without any right or wrong suggestions. She reported that a large majority wrote perfect solutions. Without doubt, the college students are a great deal more practiced in the mechanics and strategy of multiple-choice examinations.

Intelligence quotients. Scores on the California Test of Mental Maturity are available for 176 of the students who began logic in 1961. The lower quartile ranges from 98 to 122, t[illegible] second quartile 123-130, the third quartile 130-142, and the fourth quartile 142-184.

Subjective evaluations by teachers. In June of 1962 the teachers in the program were asked for subjective evaluations of general benefits to their students that they would attribute to the students' study of logic. They were given two leading questions:

I. What values can you attribute to the program? Include frank comments of your own and comments you have heard from others.

The following are the replies we received:

1. "Reading growth: the class median growth showed a gain of more than two grade points as shown by Iowa tests. Confidence in meeting new symbolisms or new systems of notation was realized."

2. "Enthusiasm of the students for the program and support of the parents showed that the subject matter is interesting to above-average students. It offered good experience in vocabulary building by a greater awareness of language and meanings. Students began to challenge careless statements by the teacher.

"A more precise way of attacking problems was acquired by some of the students. A lively interest appeared in solving problems in an original, individual kind of way on the part of some students. The

excitement of a new subject in the curriculum at this grade level stimulated the students."

3. "I have thoroughly enjoyed teaching this first year of mathematical logic. It has been a challenge to me and it has been a rewarding experience to see children of this age respond intelligently and enthusiastically to a whole new way of thinking.

"This experience has not been, however, a carefree adventure. Aside from the fact that it was a new subject for me to present, the time-consuming job of reviewing each day's assignments was sometimes a great burden. This was particularly true when we reached long proofs where the individuality of approach to solving problems became apparent. Absences of two or more days often required coaching outside of class time. The suggestion that we use some of the abler students to assist the teacher was a good one and helped to some extent, but not greatly.

"In spite of these reservations I am looking forward to going on next year.

"I think it is interesting to note some of the results of this last test. Two of the outstanding students were 6th graders, but the low scorers were also 6th graders. The 5th graders generally did much better. I have tried to analyze this difference and have concluded that it may be attributed to a difference in attitude toward this subject and possibly to other subjects. I discussed some of these children with their home-room teacher and learned that, with one exception, the low, low scorers in math logic were not performing up to their ability in other subjects. Another thing that may be considered in evaluating this class is the richness of this last year's curriculum. In the field of math alone they were exposed to new concepts not usually encountered before 8th grade and high school--and so in every other subject. Therefore, their interest was not centered in math logic.

"On the other hand, in Room 12, composed of the more able students of the two fifth grades, math logic was the new subject introduced into the regular 5th grade curriculum. They were very proud to be studying it, and most of them worked diligently. In this last test, their median

was seven points less than the other class of higher ability and their average only 2.3 points lower."

4. "The program develops creative and critical thinking, responsibility for homework; provides challenge and stimulation to the advanced child; gives prestige in home and school to children who are not always as socially accepted as others."

5. "There were no adverse comments from parents. They appeared to be flattered and curious as to the outcome. I think it made the pupils aware of the fact that there are other areas of learning than routine arithmetic, spelling, and reading."

6. "Parents are impressed and interested. I enjoy teaching the program also."

II. Have you signs of transfer of either attitudes, information, or ways of attacking problems?

We received these replies:

1. "Not very much. I particularly stressed transfer when discussing errata and students brought examples all year long. There was greater language awareness."

2. "I found enhanced self-confidence in achieving a mastery of a kind of discipline that some students originally thought could not be mastered by them. They acquired a more critical attitude toward statements they ran across; a recognition of the need of analysis in complex situations; and an awareness of sentence structure in written language. Students who at the beginning seemed to have difficulties came through with good work eventually."

3. "Signs of transfer were more apparent among the children with exceptional ability. They were very apparent in arithmetic. One child remarked that a section of his Reading Achievement Test was made easier because of the methods of logical interpretation he had learned. There was also some frustration in their trying to apply their knowledge or methods in other fields."

4. "Those pupils who were more successful in logic were better students in other class subjects. They were determined and tenacious in arithmetic. They were more inquiring and more analytical in their conclusions."

5. "There has been a transfer in arithmetic. One helps the other. There is also a growing awareness in English grammar as the two are compared."

9. Conclusions and implications.

Our summary conclusions are the following.

1. The upper quartile of elementary school students can achieve a significant conceptual and technical mastery of elementary mathematical logic. The level of mastery is 85 to 90 percent of that achieved by comparable university students.

2. This mastery of the subject matter can be accomplished in an amount of time comparable to that needed by college students, but allocated over a much longer period of time and with considerably more direct teacher supervision.

3. The more dedicated and able elementary-school teachers can be given sufficient training in five or six semester hours of instruction to teach classes in elementary mathematical logic. It is probably essential that a teacher training program of no more than six semester hours be very closely geared to the actual program of instruction the teacher will follow in his or her own classroom.

4. Anecdotal evidence from teachers suggests that there is some carry-over in critical thinking and attitude into other fields, especially arithmetic, reading and English. More explicit behavioral data on this point would be desirable.

5. The work with the special summer class in 1963 indicates that able elementary-school students who have received prior training in mathematical logic can make rapid progress in other parts of modern mathematics organized on a deductive basis. We hope to be able to continue this line of investigation in the future and to determine to what extent the deductive methodology and viewpoint of modern mathematics may be made a natural and intuitive part of the young student's mathematical knowledge.

10. References

(1) Hill, S. A. A study of the logical abilities of children. Unpublished doctoral dissertation, Stanford University, 1960.

(2) Hyram, G. H. An experiment in developing critical thinking in children. J. exp. Educ., 1957, 26, 125-132.

(3) Inhelder, B., Piaget, J. The Growth of Logical Thinking. trans. A. Parsons and S. Milgram, New York: Basic Books, Inc., 1958.

11. Appendices.

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A. Logic Test I for 1962-1963, 12 pages	
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Logic Test no. 1

Follow directions on the answer sheet. Put your answers by the same number as used on these problem sheets. In numbering questions, many numbers were omitted to make the numbering of each section of the test start at the top or center of the answer sheet.

A. Using

P = 'It is raini. Q = 'The ground is wet'

S = 'The sun is shining'

select the best translation of each of the following English sentences:

1. The sun is not shining.
   1. S
   2. -P
   3. -S
   4. -(-S)
   5. P

2. Either it is raining or the ground is not wet.
   1. -(P & Q)
   2. P v -S
   3. -Q v P
   4. S v -Q
   5. P v -(Q)

3. If the sun is shining then it is not raining.
   1. -P → S
   2. S → -P
   3. Q → P
   4. -(S → P)
   5. -Q → S

4. It is not the case that either it is raining or the sun is shining.
   1. -P v S
   2. Q & -S
   3. -(P → S)
   4. -Q v P
   5. -(P v S)

5. Both either the sun is not shining or it is raining and the ground is wet.

1. (-S v P) & Q
2. -(S v P) & Q
3. -S v (P & Q)
4. -S v P & Q
5. -(S) v (P & -Q)

6. If it is not raining and the sun is shining then the ground is not wet.

1. -(P & S) → -Q
2. (-P & S) → -Q
3. -P → S & -Q
4. -P & (S → -Q)
5. P & -S → -Q

7. It is not the case that either the sun is shining or if it is not raining then the ground is wet.

1. -(S v -P) → Q
2. -S v -(P → Q)
3. -(S v (-P → Q))
4. -S v (-P → Q)
5. -(S v -(P → Q))

B. Find a negation of each of the following.

16. R

1. -(R)
2. --(-R)
3. --R
4. -(-(-(-(-(R)))))
5. -(-(-R))

17. -R

1. -(R)
2. --(-R)
3. --R
4. -(-(-(-(-(R)))))
5. -(-(-R))

18. --(-R)

1. -(R)
2. --(-R)
3. --R
4. -(-(-(-(-(R)))))
5. -(-(-R))

19. B & C

1. -B & -C
2. B & C
3. -(B & C)
4. -(-B & -C)
5. -B & C

20. -(B & C)

1. -B & -C
2. B & C
3. -(B & C)
4. -(-B & -C)
5. -B & C

21. R → -S

1. R → S
2. -R → S
3. -(R → S)
4. -R →--S
5. -(R → -S)

22. -A ↔ B

1. A ↔ B
2. -(A ↔ B)
3. -(A ↔ -B)
4. -(-A ↔ B)
5. A ↔ -B

23. x≠y v y≮z

1. x=y & y<z
2. x=y v y<z
3. -(x≠y v y≮z)
4. -(x=y v y≮z)
5. -(x=y v y<z)

C. For each of problems 31 through 41, decide which of the five choices given here best describes the string of symbols.

1. a conjunction
2. a disjunction
3. a conditional
4. a biconditional
5. none of those listed in 1 - 4

31. (P v Q) & -R
32. -S → T v U
33. -(A v (B & C))
34. x=y → x<5 ↔ y<5
35. -D v E & -F
36. -(x<y & x>z)
37. P & -Q → S v T
38. -((A → B) & C) v D
39. L v -M → N ↔ P & -Q
40. S v (T → -C) & -(P ↔ Q)
41. S v (T → B -C)

D. In the following brief arguments, premises are above the lines, conclusions below. Select the rule that justifies the conclusion if there is one given. If it is a bad argument, or the rule is not among the given choices, mark 5, no justifying rule.

46. $A \rightarrow B$
$-B$
$-A$

1. PP
2. TT
3. TP
4. HS
5. no justifying rule

47. $S \rightarrow R$
$-S$
$-R$

1. PP
2. TT
3. TP
4. DN
5. no justifying rule

48. Q
$-Q$

1. DN
2. S
3. A
4. LA
5. no justifying rule

49. $x<y \rightarrow y>3$
$x<y$
$y>3$

1. PP
2. TT
3. TP
4. S
5. no justifying rule

50. $-C \rightarrow D$
$-C$
D

1. S
2. TT
3. TP
4. HS
5. no justifying rule

51. $A \rightarrow -B$
$-B$
$-A$

1. PP
2. TT
3. TP
4. DN
5. no justifying rule

52. P v -Q
-P

-Q

1. PP
2. TT
3. TP
4. S
5. no justifying rule

53. x≤y & y≠6
---
y≠6

1. PP
2. TP
3. S
4. A
5. no justifying rule

54. -C → -D
--D

-C

1. PP
2. TT
3. TP
4. DN
5. no justifying rule

55. -P → -S
--S

P

1. PP
2. TT
3. TP
4. DN
5. no justifying rule

56. A v C

A

1. S
2. LA
3. DP
4. DS
5. no justifying rule

57. -(-P v Q)

P & -Q

1. DS
2. DN
3. A
4. DL
5. no justifying rule

58. -P v Q

-Q v P

1. S
2. CL
3. TP
4. DL
5. no justifying rule

59. C v D
-B

-B & (C v D)

1. TP
2. A
3. LA
4. PP
5. no justifying rule

60. $\underline{y \not< 5}$

$y = x \vee y \not< 5$

1. DP
2. TP
3. A
4. LA
5. no justifying rule

61. $Q \rightarrow P$

$\underline{-P \rightarrow S}$

$Q \rightarrow S$

1. HS
2. PP
3. TT
4. DS
5. no justifying rule

62. $-A \rightarrow B$

$J \rightarrow -K$

$\underline{J \vee -A}$

$-K \vee B$

1. PP
2. HS
3. TP
4. DS
5. no justifying rule

63. $\underline{--(x+y=3 \ \& \ y=1)}$

$x+y=3 \ \& \ y=1$

1. DL
2. DS
3. DN
4. DP
5. no justifying rule

64. $\underline{-H \vee -H}$

$-H$

1. DP
2. TP
3. S
4. LA
5. no justifying rule

65. $-(R \ \& \ -Q) \rightarrow S$

$\underline{S \rightarrow (-Q \vee T)}$

$-(R \ \& \ -Q) \rightarrow -Q \vee T$

1. LA
2. HS
3. TP
4. DS
5. no justifying rule

66. $A \leftrightarrow -B$

$\underline{-B \vee -A}$

$-B \rightarrow A$

1. DS
2. TP
3. LB
4. TT
5. no justifying rule

E. Answer or complete the following by selecting for each the choice that best fills in the blank.

76. Any sentence containing a sentential connective is called _____ sentence.

1. a molecular
2. a dominant
3. an atomic
4. a derived
5. a consequent

77. A set of sentences as premises and a conclusion derived from these sentences is called _____.

1. a proof
2. a translation
3. an argument
4. a derivation
5. dominant

78. The dominant connective of a molecular sentence _____.

1. is always an arrow.
2. governs the sentence as a whole.
3. is determined entirely by parentheses.
4. is the first one.
5. is always a double arrow.

79. A sentence without any logical connective is _____.

1. not a logical sentence
2. a molecular sentence
3. a premise
4. an atomic sentence
5. not a formula

80. A series of sentences or steps, in which every step is either a premise or is derived directly from the steps that go before it by a definite rule is _____.

1. an argument
2. consistent
3. a formal proof
4. invalid
5. a conditional

81. A sentence in the form, ( ) v ( ), is _____.

1. a conjunction
2. a disjunction
3. a conditional
4. atomic
5. a conclusion

82. A sentence in the form, $P \rightarrow Q$, is a _____.

1. consequent
2. conditional
3. connective
4. biconditional
5. antecedent

83. The P in $P \rightarrow Q$ is called the _____.

1. consequent
2. conditional
3. connective
4. biconditional
5. antecedent

84. The Q in $P \rightarrow Q$ is called the _____.

1. consequent
2. conditional
3. connective
4. biconditional
5. antecedent

85. A formula in the form, -( ) , is _____.

1. simple
2. a disjunction
3. atomic
4. a conditional
5. a negation

86. A sentence in the form ( ) & ( ) is a _____.

1. conditional
2. consequent
3. contradiction
4. conjunction
5. conclusion

87. A sentence in the form P ↔ Q is a _____.

1. biconditional
2. disjunction
3. conditional
4. true one
5. atomic sentence

F. In each of the following (91 - 94) select the appropriate <u>first</u> step beyond the premises to most efficiently lead to the conclusion.

91. <u>Prove: P & Q</u>

1. P
2. R v Q
3. P → -R

(1) P → Q
(2) -R
(3) Q
(4) P & Q
(5) -R → Q

92. <u>Prove: P v S</u>

1. P → -S
2. -S → R
3. V → -R
4. V

(1) -(-R → V)
(2) P → -R
(3) -R
(4) S
(5) V → -R

93. Prove: P → Q

1. P	(1) -T v R
2. R → -S	(2) R → -T
3. -(P → Q) → S	(3) -S
4. T → -(P → Q)	(4) -S v -(P → Q)
5. -S → -T	(5) -T
6. R	
7. P → -T v R	

94. Prove: x+y=1

1. x+y=2 v x+y=3	(1) xy≠0 v x/y≠tanθ
2. x+y=2 → xy≠0	(2) x+y=2 → x/y=tanθ
3. x+y=3 → x/y≠tanθ	(3) x+y=3 → x/y=tanθ
4. xy≠0 → x+y=1	(4) x+y=1 → x+y=3
5. x/y≠tanθ → x+y=1	(5) xy>0 → x+y≠2

G. In the formal derivations for problems (a), (b), and (c), select the justification for each line from among the choices given opposite its bracket. Lines are numbered for the answer sheet.

(a)

( )	A v B	P
( )	(A & C) → (G & H)	P
( )	-(B v -D)	P
( )	D → S	P
(106)	-B & D	1. TT
(107)	-B	2. S
(108)	A	3. A
		4. DL
		5. TP
(109)	D	1. A
(110)	C	2. TP
(111)	A & C	3. PP
		4. DP
		5. S

(continued on next page)

(112)	G & H		1. DL
(113)	H & G		2. LA
(114)	(H & G) v E		3. DS
			4. PP
			5. CL

(b)

( )	$x<y \rightarrow y>4$	P
( )	$y\not>4$ v $z=y$	P
( )	$x\not<3 \rightarrow y>4$	P
( )	$x<y$ v $x\not<3$	P
(121)	$y>4$ v $y>4$	1. TP
(122)	$y>4$	2. HS
(123)	$z=y$	3. DP
		4. DS
		5. PP
(124)	$y>4$ & $z=y$	1. DN
(125)	$-(y\not>4$ v $z\neq y)$	2. DL
		3. DS
		4. DP
		5. A

(c) ( ) -(-S v N)	P	
( ) -N → Q	P	
( ) P → -Q	P	
( ) -S v -R	P	
(136) S & -N	}	{ 1. DL
(137) S	}	{ 2. HS
	}	{ 3. DS
	}	{ 4. TP
	}	{ 5. S
(138) -R	}	{ 1. PP
(139) -N	}	{ 2. TT
(140) Q	}	{ 3. TP
	}	{ 4. DP
	}	{ 5. S
(141) -P	}	{ 1. A
(142) -R & -P	}	{ 2. LA
(143) -(R v --P)	}	{ 3. DL
	}	{ 4. DN
	}	{ 5. TT

Logic Test no. 2

Follow directions on the answer sheet. Put your answers by the same number as used on these problem sheets. In numbering questions, many numbers were omitted to make the numbering of each section of the test start at the top or center of the answer sheet.

A. In the formal derivations for problems (a), (b), and (c) select the justifying rule for each line from among the choices given in the bracket opposite its bracket. Lines are numbered for the answer sheet.

(a)

( ) $A \rightarrow (P \rightarrow \sim F)$ P

( ) $P \longleftrightarrow Q$ P

(1) A
(2) $P \rightarrow \sim F$

1. TT
2. PP
3. CP
4. P
5. LB

(3) $Q \rightarrow P$
(4) $Q \rightarrow \sim F$

1. LB
2. CP
3. PP
4. HS
5. TP

(5) $\sim(\sim Q \vee \sim F)$
(6) $Q \,\&\, F$

1. DS
2. CL
3. A
4. DL
5. P

(7) F
(8) Q

1. A
2. S
3. TT
4. TP
5. PP

(9) $\sim Q$
(10) $Q \,\&\, \sim Q$

1. TT
2. RAA
3. LA
4. A
5. DN

(11) $\sim Q \vee \sim F$
(12) $A \rightarrow \sim Q \vee \sim F$

1. CP
2. DL
3. RAA
4. LB
5. P

(b)

( )	$(x<y \rightarrow y>5)$ & $y=6$		P
( )	$x>5$ v $x<y$		P
( )	$x>5 \rightarrow x=y$		P
(16)	$y \not> 5$	}	{ 1. TT; 2. P; 3. HS; 4. S; 5. CP
(17)	$x<y \rightarrow y>5$		
(18)	$y>5$ v $x=y$	}	{ 1. S; 2. LA; 3. PP; 4. TP; 5. DS
(19)	$x=y$		
(20)	$y=6$		
(21)	$x=y$ & $y=6$	}	{ 1. CL; 2. CP; 3. A; 4. DL; 5. DP
(22)	$y \not> 5 \rightarrow x=y$ & $y=6$		

(c)

( )	D v H		P
( )	-A → -H		P
( )	-(D v B)		P
(31)	-D & -B	}	{ 1. S; 2. TP; 3. CL; 4. DL; 5. LA
(32)	-D		
(33)	H	}	{ 1. P; 2. TT; 3. TP; 4. CP; 5. HS
(34)	A		
(35)	A → B		
(36)	B	}	{ 1. DP; 2. S; 3. DN; 4. PP; 5. TP
(37)	-B		
(38)	B & -B	}	{ 1. RAA; 2. CL; 3. LB; 4. DS; 5. A
(39)	-(A → B)		

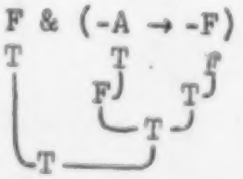
B. For each of the following sets or premises, if it is consistent, find a set of truth assignments for the atomic sentences that would prove they are consistent. If the set is inconsistent, choose (5) inconsistent. Watch the numbering or the problems. Put your answer opposite the same number on the answer sheet.

	Premises	Answers
(46)	-A → B -C → B -B v -D C & D	1. T | F: C | A; D | B — 2. T | F: B | A; D | C — 3. T | F: A | B; C; D — 4. T | F: B | A; C; D — 5. inconsistent
(49)	P & -Q B → Q A v B A → Q	1. T | F: P | B; Q; A — 2. T | F: P | A; Q | B — 3. T | F: A; B; P; Q — 4. T | F: P | Q; B; A — 5. inconsistent
(52)	-P S v R Q & -R P v Q	1. T | F: P | R; Q; S — 2. T | F: Q | P; S | R — 3. T | F: P | S; Q; R — 4. T | F: Q | P; R; S — 5. inconsistent
(55)	J v K L → -J L & N N → -K	1. T | F: J | L; K | N — 2. T | F: L | J; K; N — 3. T | F: L | J; N; K — 4. T | F: L | K; J; N — 5. inconsistent

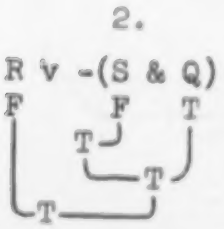
C. Pick out the correct truth diagram.

(61)

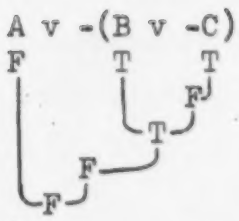
1.



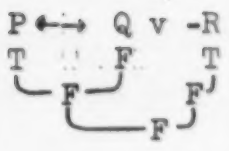
2.



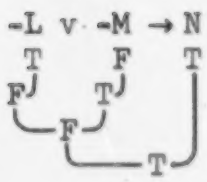
3.


A v -(B v -C)
F T T
F
T
F
F

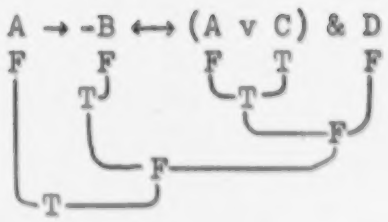
4.


P ↔ Q v -R
T F T
F
F
F

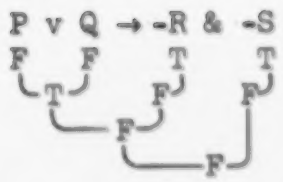
5.


-L v -M → N
T F T
F
T
F
T

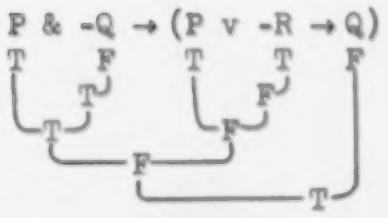
(64) 1.


A → -B ↔ (A v C) & D
F F F T F
T
T
F
F
T

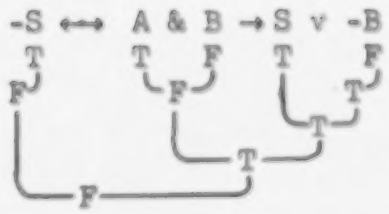
2.


P v Q → -R & -S
F F T T
T
F
F
F

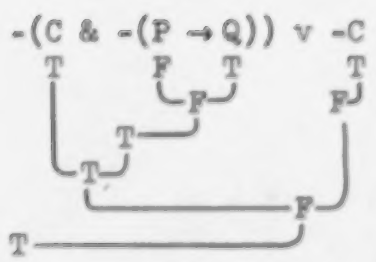
3.


P & -Q → (P v -R → Q)
T F T T F
T
T
F
F
T

4.


-S ↔ A & B → S v -B
T T F T F
F
F
T
T
F

5.


-(C & -(P → Q)) v -C
T F T T
F
F
T
T
F
T

D. For each of the following arguments, if it is valid, answer 1. (valid). If it is invalid, find the set of truth assignments for the atomic sentences that would prove its invalidity.

(76) Prove: $Q \rightarrow P$
(1) $-Q \rightarrow S$
(2) $-S \;\&\; Q$
(3) $P \vee -S$

1. valid

2.

T	F
Q	S
P	

3.

T	F
	Q
	S
	P

4.

T	F
S	Q
	P

5.

T	F
Q	P
	S

(79) Prove: K
(1) $J \vee K$
(2) $-(N \vee -L)$
(3) $J \leftrightarrow -L$

1. valid

2.

T	F
J	K
	L
	N

3.

T	F
K	N
L	J

4.

T	F
N	K
	J
	L

5.

T	F
J	
K	
L	
M	

(82) Prove: $-L$
(1) $J \vee K$
(2) $-(N \vee -L)$
(3) $J \leftrightarrow -L$

1. valid

2.

T	F
L	K
J	
N	

3.

T	F
L	J
K	N

4.

T	F
J	L
K	N

5.

T	F
N	K
	J
	L

E. (91) Which one of the following is not a tautology? (Hint: you may need to make out truth tables.)

1. $A \vee (B \rightarrow -A)$
2. $(A \;\&\; -B) \vee (A \rightarrow B)$
3. $A \rightarrow (-A \rightarrow B)$
4. $A \vee -B \leftrightarrow -A$
5. $-(A \;\&\; -(B \vee A))$

(95) Which one of the following five rules is invalid? (Hint: you many need to make out truth tables.)

1. From $-(P \rightarrow Q)$ conclude $P \;\&\; -Q$.
2. From $P \rightarrow Q$ and $-P$ conclude $-Q$.
3. From $P \rightarrow -P$ conclude $-P$.
4. From $-(-P \;\&\; Q)$ and Q conclude P.
5. From $-P$ conclude $P \rightarrow Q$.

F. For each of the following answer:

1. if it is a term
2. if it is a predicate
3. if it is an atomic formula but not an atomic sentence
4. if it is an atomic sentence
5. if it is none of the above

(106) Alice fell slowly down a deep hole

(107) About 18 years old

(108) 7 + 5

(109) plays with Bill

(110) Mary is not bashful

(111) x is old

(112) x is an odd number

(113) without any help

(114) y

(115) is a coward

(116) the first astronaut

(117) Three plus five is greater than six

(118) thinks he can learn logic without studying

(119) Rz

(120) If Harry is sick then Jim delivers his papers.

Find the best translation into logical symbols for each of the following. Terms and predicates are symbolized by obvious initial letters. Be careful to number correctly on the answer sheet.

(121) John has red hair.

1. $(\forall x)(x=j \ \& \ Rx)$
2. Jr
3. $x=j \rightarrow Rj$
4. Rj
5. $(\forall x)(Rx \rightarrow x=j)$

(122) Everything is new.

1. $(\forall x)(Ex \rightarrow Nx)$
2. Ne
3. $(\forall x)(Nx)$
4. $(\forall x)(Ne)$
5. $(\forall x)(Ex \ \& \ Nx)$

(123) - That spilt milk is not worth crying over.

1. -Ws
2. -(∀x)(Sx → Wx)
3. (∀x)(-Wx → Sx)
4. (∀x)(Wx → Sx)
5. -Sw

(124) Bears hibernate in the winter.

1. Hb
2. (∀x)(Bx → Hx)
3. (∀x)(Hx → Bx)
4. (∀x)(x=b → Hx)
5. (x)(-Bx → -Hx)

(125) Nothing is in this bottle.

1. -(∀x)(Bx)
2. Bn
3. (∀x)(Nx → Bx)
4. (∀x)(Bx → -Ax)
5. (∀x)(-Bx)

(126) The product of three and six is even

1. (∀x)(x = P_3 & P_6 → Ex)
2. (∀x)(x = $P_{3,6}$ → Ex)
3. E (3·6)
4. (∀x)(Ex → x = 3·6)
5. E (3·6)→ P (3·6)

(127) Not all dogs have hair.

1. -(∀x)(Dx → Hx)
2. (∀x)(Dx → -Hx)
3. (∀x)(Dx → Hx)
4. (∀x)(-Dx → Hx)
5. -(∀x)(-Hx → Dx)

(128) A rolling stone gathers no moss.

1. Rs → -Gm
2. -(∀x)(Rx → Gx)
3. -Gr
4. (∀x)-(Rx → Gx)
5. (∀x)(Rx → -Gx)

(129) Billy loves his sisters.

1. Lbs
2. (∀x)(Sxb → Lxb)
3. (∀x)(Lbx → Sxb)
4. (∀x)(Sxb → Lbx)
5. (∀x)(Lxb → Sxb)

(130) No turtles run races.

1. (∀x)-(Tx → Rx)
2. -(∀x)(Tx → Rx)
3. (∀x)(Tx → -Rx)
4. -Rt
5. -Tr

(131) Only cheetahs run 80 mph.

1. $(\forall x)(Cx \rightarrow Rx)$
2. $-(\forall x)(Cx \rightarrow Rx)$
3. $(\forall x)(Rx \rightarrow Cx)$
4. $(\forall x)(-Cx \rightarrow Rx)$
5. Rc

G. Complete the formal proof of each of the following derivations by selecting one of the alternatives for each of the lines after the premises. The derived lines are numbered to show where on the answer sheet to put the answer. The premise lines are numbered for reference only.

Prove: $6 + (4+2) = 12$

(10)	$2 \cdot 6 = 12$	P
(11)	$(\forall x)(x+x = 2 \cdot x)$	P
(12)	$4+2 = 5+1$	P
(13)	$6 = 5+1$	P
(136)	1. $2+2 = 4$	2/x 11
	2. $6 + (4+2) = 6 + (4+2)$	L
	3. $6 + (4+2) = 2 \cdot 6$	I 10, 11
	4. $2 \cdot 6 = 2 \cdot (4+2)$	I 10, 12
	5. $6 + 6 = 12$	6/x 11
(137)	1. $6 + (5+1) = 12$	I 136, 13
	2. $6 + 6 = 2 \cdot 6$	6/x 11
	3. $6 + (4+2) = 12$	I 136, 10
	4. $6 + (4+2) = 6 + (5+1)$	I 136, 12
	5. $6 + (5+1) = 6 + (5+1)$	L
(138)	1. $6 + (4+2) = 6 + 6$	I 137, 13
	2. $6 + (4+2) = 12$	I 136, 137
	3. $6 + (4+2) = 6 + 6$	I 136, 137
	4. $6 + (4+2) = 6 + (5+1)$	I 137, 12
	5. The proof is completed on line (137)	

(139) 1. 6 + (4+2) = 12 I 137, 138
2. 6 + 6 = 2·6 6/x 11
3. 6 + (4+2) = 6 + 6 I 138, 13
4. 6 + (5+1) = 2·6 I 136, 10
5. The proof is completed on an earlier line.

(140) 1. 6 + (4+2) = 12 I 139, 136
2. 6 + (4+2) = 2·6 I 138, 139
3. 6 + (5+1) = 12 I 139, 10
4. 6 + (4+2) = 12 I 138, 139
5. The proof is completed on an earlier line.

(141) 1. 6 + (4+2) = 12 I 139, 140
2. 6 + (4+2) = 12 I 140, 12
3. 6 + (4+2) = 12 I 140, 10
4. The argument is invalid.
5. The proof is completed on an earlier line.

Prove: -(Rb v Tb)

(10) (∀x)(Sx & -Px) P
(11) Sb → -Rb P
(12) Pa v -Ta P
(13) a = b P

(142) 1. Sb → -Ra I 11, 13
2. Sb & -Pa b/x, a/x, 1
3. Pb v Ta I 12, 13
4. -Rb S 11
5. Sb & -Pb b/x 10

(143)	1.	Sa & -Pa	n/x 10
	2.	Sb	S 142
	3.	-Ta	DP 12
	4.	Sb & -Pb	S 142
	5.	-Sb	TT 11, 142
(144)	1.	-Rb & -Ta	A 142, 143
	2.	Sa	CP 142, 142
	3.	-Pb	S 143
	4.	-Rb	PP 11, 143
	5.	Ra	PP 142, 143
(145)	1.	Ta	TP 142, 144
	2.	-Pb	S 142
	3.	-Rb & -Tb	I 144, 13
	4.	-Pa	S 142
	5.	-Rb v -Ta	DS 10, 11, 12
(146)	1.	-Rb v -Tb	I 145, 13
	2.	-Ta	TP 12, 145
	3.	-Pa	I 145, 13
	4.	Tb	I 145, 13
	5.	-(Rb v Tb)	DL 145
(147)	1.	-Ta	TP 12, 146
	2.	-Tb	I 146, 13
	3.	Rb v Tb	LA 146
	4.	-(Rb v Tb)	DL 146
	5.	The proof is completed on line (146).	

(148) 1. -(Rb v Tb) DN 147
2. -Tb I 147, 13
3. Ta → Tb CP 145, 13
4. -Rb & -Tb A 144, 147
5. The proof is completed on an earlier line.

(149) 1. -(Rb v Tb) DL 148
2. Rb v Tb DS 12, 11, 148
3. -Rb & -Tb A 144, 148
4. -(Rb v Tb) DN 148
5. The proof is completed on an earlier line.

(150) 1. -(-Rb & -Tb) DL 149
2. -(Rb v Tb) DN 149
3. -(Rb v Tb) DL 149
4. The proof requires two more steps.
5. The proof is completed on an earlier line.

Algebra of Sets

Axioms

Ax. 1. $A \cup \Phi = A$

Ax. 2. $A \cap X = A$

Ax. 3. $A \cup B = B \cup A$

Ax. 4. $A \cap B = B \cap A$

Ax. 5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Ax. 6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Ax. 7. $A \cup {\sim}A = X$

Ax. 8. $A \cap {\sim}A = \Phi$

Ax. 9. $X \neq \Phi$

Theorems

Th. 1. $A \cup A = A$

Th. 2. $A \cap A = A$

Th. 3. $A \cup X = X$

Th. 4. $A \cap \Phi = \Phi$

Th. 5. ${\sim\sim}A = A$

Th. 6. $A = {\sim}B \rightarrow B = {\sim}A$ Th. 6A. ${\sim}A = {\sim}B \rightarrow A = B$

Th. 7. $A \cup B \neq \Phi \rightarrow A \neq \Phi \vee B \neq \Phi$

Th. 8. $A \cap B \neq \Phi \rightarrow A \neq \Phi$

Th. 9. $A \cup (A \cap B) = A$

Th. 10. $A \cap (A \cup B) = A$

Th. 11. $A \cup B = X \ \& \ A \cap B = \Phi \rightarrow B = {\sim}A$

Th. 12. $A \cup ({\sim}A \cup B) = X$

Th. 13. $A \cap ({\sim}A \cap B) = \Phi$

Th. 14. $A \cup B = {\sim}({\sim}A \cap {\sim}B)$

Th. 15. $A \cap B = {\sim}({\sim}A \cup {\sim}B)$

Th. 16. $(A \cup B) \cup C = A \cup (B \cup C)$

Th. 17. $(A \cap B) \cap C = A \cap (B \cap C)$

Th. 18. $A \neq {\sim}A$

Definition 1. $A \sim B = A \cap \sim B$

Th. 19. $A \sim A = \Phi$

Th. 20. $A \sim (A \cap B) = A \sim B$

Th. 21. $A \cap (A \sim B) = A \sim B$

Th. 22. $(A \sim B) \sim A = \Phi$

Th. 23. $(A \sim B) \cup B = A \cup B$

Th. 24. $(A \cup B) \sim B = A \sim B$

Definition 2. $A \subseteq B \leftrightarrow A \cup B = B$

Th. 25. $A \subseteq A \cup B$

Th. 26. $(A \sim B) \subseteq A$

Th. 27. $A \subseteq A$

Th. 28. $A \subseteq B \,\&\, B \subseteq A \rightarrow A = B$

Th. 29. $\Phi \subseteq A$

Th. 30. $A \subseteq \Phi \rightarrow A = \Phi$

Th. 31. $A \subseteq B \leftrightarrow \sim B \subseteq \sim A$

Th. 32. $A \subseteq B \leftrightarrow A \cap B = A$

Th. 33. $A \subseteq B \,\&\, B \subseteq C \rightarrow A \subseteq C$

Definition 3. $A \subset B \leftrightarrow A \subseteq B \,\&\, A \neq B$

Th. 34. $\neg(A \subset A)$

Th. 35. $A \subset B \rightarrow \neg(B \subset A)$

Th. 36. $A \subset B \,\&\, B \subset C \rightarrow A \subset C$

Th. 37. $A \subset B \rightarrow A \subseteq B$

Th. 38. $A \cap B \subset A \rightarrow A \neq B$

Th. 39. $A \cap B \neq \Phi \rightarrow A \sim B \subset A$

Th. 40. $A \subseteq C \,\&\, B \subseteq C \rightarrow A \cap B \subseteq C$

Th. 41. $A \subseteq C \,\&\, A \subseteq B \rightarrow A \subseteq B \cap C$

Axioms of Topology

Ax. 10. $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Ax. 11. $A \subseteq \bar{A}$

Ax. 12. $\bar{\bar{A}} = \bar{A}$

Ax. 13. $\bar{\Phi} = \Phi$

Th. 42. $A \subseteq B \rightarrow \overline{A} \subseteq \overline{B}$

Th. 43. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Definition 4. A is closed $\leftrightarrow \overline{A} = A$

Definition 5. A is an open set $\leftrightarrow \sim A$ is closed

$\leftrightarrow \overline{\sim A} = \sim A$

$\leftrightarrow A = \sim\overline{\sim A}$

Th. 44. $\overline{A} \sim \overline{B} \subseteq \overline{A \sim B}$

Th. 45. $X = \overline{X}$

Th. 46. If A and B are closed, then their union is closed:

$$\overline{A} = A \ \& \ \overline{B} = B \rightarrow A \cup B = \overline{A \cup B}$$

Th. 47. $\overline{A} = A \ \& \ \overline{B} = B \rightarrow \overline{A \cap B} = A \cap B$

Th. 48. If A and B are open, then their union is open:

$$\overline{\sim A} = \sim A \ \& \ \overline{\sim B} = \sim B \rightarrow \overline{\sim(A \cup B)} = \sim(A \cup B)$$

Th. 49. If A and B are open, then their intersection is open:

$$\overline{\sim A} = \sim A \ \& \ \overline{\sim B} = \sim B \rightarrow \overline{\sim(A \cap B)} = \sim(A \cap B)$$

Th. 50. The empty set is both open and closed.

Th. 51. The universe is both open and closed.

Th. 52. If G is open then $G \cap \overline{A} \subseteq \overline{G \cap A}$

Definition 6. Boundary: $\mathrm{Bd}(A) = \overline{A} \cap \overline{\sim A}$

Definition 7. Interior: $\mathrm{Int}(A) = \sim\overline{\sim A}$

Th. 53. $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$

Th. 54. $A \subseteq B \rightarrow \mathrm{Int}(A) \subseteq \mathrm{Int}(B)$

Th. 55. $\mathrm{Int}(A) \cup \mathrm{Int}(B) \subseteq \mathrm{Int}(A \cup B)$

Th. 56. $\mathrm{Int}(A) = A \sim \overline{\sim A}$

Th. 57. $\mathrm{Int}(A) = A \sim \mathrm{Bd}(A)$

Th. 58. $\mathrm{Int}(A) \subseteq A$

Th. 59: $\mathrm{Bd}(\sim A) = \mathrm{Bd}(A)$

Th. 60. $\mathrm{Bd}(\overline{A}) \subseteq \mathrm{Bd}(A)$

Th. 61. $\mathrm{Bd}(A) = (A \cap \overline{\sim A}) \cup (\overline{A} \sim A)$

Th. 62. $A \cup \mathrm{Bd}(A) = \overline{A}$

Th. 63. $\mathrm{Int}(A) \cap \mathrm{Bd}(A) = \Phi$

Th. 64. $\mathrm{Bd}(\mathrm{Int}(A)) \subseteq \mathrm{Bd}(A)$

Appendix D. Project Staff

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